Warsaw University of Technology

Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

Example. Beam, bar, and spring assembly

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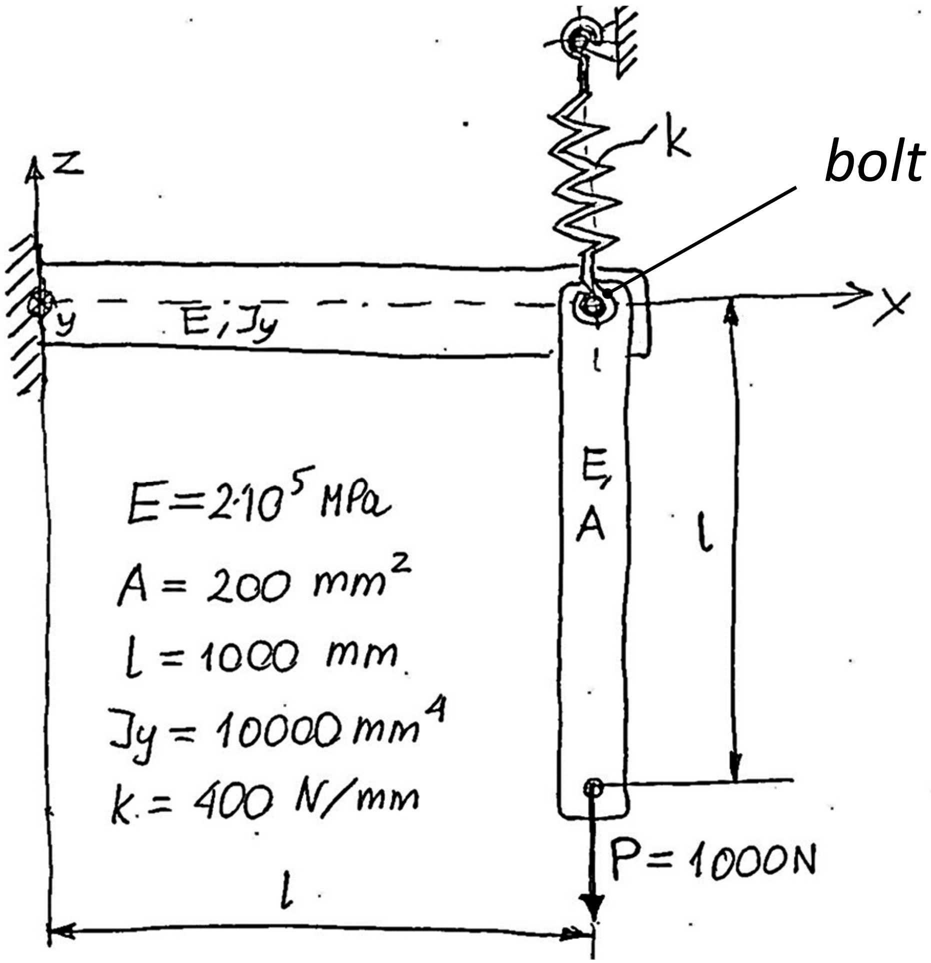
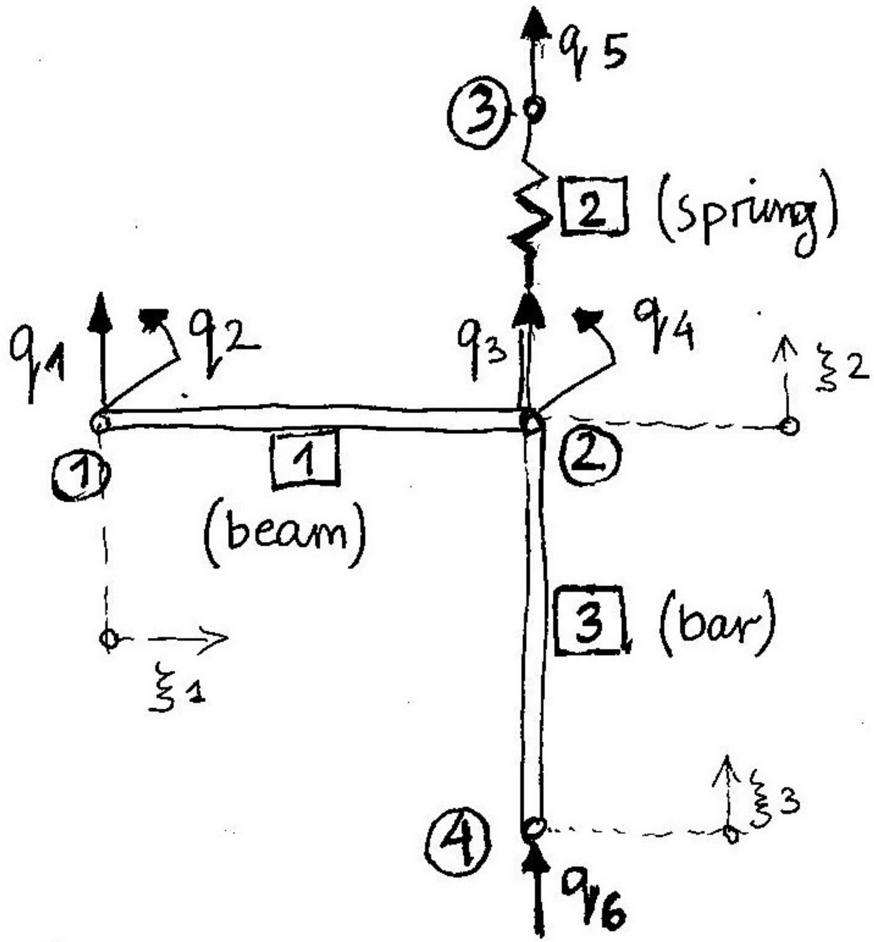
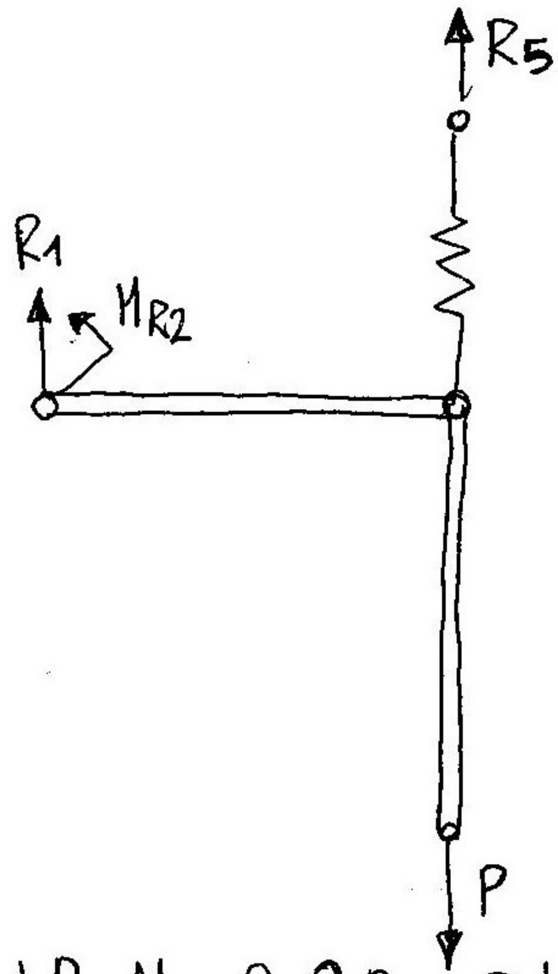
EXAMPLE: BUILD A FINITE ELEMENT MODEL OF THE STRUCTURE CONSISTING OF A BEAM, A BAR AND A SPRING. FIND UNKNOWN DISPLACEMENTS AND REACTIONS AND CHECK EQUILIBRIUM.







$$\underset{1\times 6}{\lfloor q \rfloor} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6 \rfloor \qquad \underset{1\times 6}{\lfloor F \rfloor} = \lfloor R_1, M_{R2}, 0, 0, R_5, -P \rfloor$$

element $\boxed{1}$: $\underset{1\times 4}{\lfloor q \rfloor_1} = \lfloor q_1, q_2, q_3, q_4 \rfloor_1$

$$\underset{4\times 4}{[k]_1} = \frac{2EJ_y}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} \quad ; \quad \underset{6\times 6}{[k]_1^*} = \left[\begin{array}{cccc|cc} \multicolumn{4}{c|}{\multirow{4}{*}{$\underset{4\times 4}{[k]_1}$}} & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

element $\boxed{2}$: $\underset{1\times 2}{\lfloor q \rfloor_2} = \lfloor q_3, q_5 \rfloor_2$

$$\underset{2\times 2}{[k]_2} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad ; \quad \underset{6\times 6}{[k]_2^*} = \begin{array}{c} \begin{array}{cccccc} & & (3) & & (5) & \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{k} & 0 & \boxed{-k} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-k} & 0 & \boxed{k} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ (3) \\ \\ (5) \\ \\ \end{array}$$

element $\boxed{3}$: $\underset{1\times 2}{\lfloor q \rfloor_3} = \lfloor q_6, q_3 \rfloor_3$

$$\underset{2\times 2}{[k]_3} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad ; \quad \underset{6\times 6}{[k]_3^*} = \begin{array}{c} \begin{array}{cccccc} & & (3) & & & (6) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{\frac{EA}{L}} & 0 & 0 & \boxed{-\frac{EA}{L}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-\frac{EA}{L}} & 0 & 0 & \boxed{\frac{EA}{L}} \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ (3) \\ \\ \\ (6) \end{array}$$

$$[K] = \sum_{e=1}^{3} [k]_e^{*} = \begin{bmatrix} & & & & 0 & 0 \\ & & [K]_1 & & 0 & 0 \\ & & \frac{12EJ_y}{l^3} + k + \frac{EA}{l} & & -k & -\frac{EA}{l} \\ & & & & 0 & 0 \\ 0 & 0 & -k & 0 & k & 0 \\ 0 & 0 & -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} \end{bmatrix}$$

$$\underset{6\times 6}{[K]} \cdot \underset{6\times 1}{\{q\}} = \underset{6\times 1}{\{F\}}$$

+ boundary conditions

$q_1 = 0$

$q_2 = 0$

$q_5 = 0$

$$
\begin{bmatrix}
 & & & & 0 & 0 \\
 & & & & 0 & 0 \\
 & & & & & \\
 & & & & 0 & 0 \\
0 & 0 & & 0 & & 0 \\
0 & 0 & & 0 & 0 &
\end{bmatrix}
\cdot
\begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ 0 \\ q_6 \end{Bmatrix}
=
\begin{Bmatrix} R_1 \\ M_{e2} \\ 0 \\ 0 \\ R_5 \\ -P \end{Bmatrix}
\rightarrow
\underset{3\times 3}{[K]} \cdot \underset{3\times 1}{\{q\}} = \underset{3\times 1}{\{F\}}
$$

$$
\begin{array}{c}
\mathrm{I} \\[1.5em] \mathrm{II} \\[1.5em] \mathrm{III}
\end{array}
\begin{bmatrix}
\frac{12EJ_y}{l^3} + k + \frac{EA}{l} & -\frac{6EJ_y}{l^2} & -\frac{EA}{l} \\[0.5em]
-\frac{6EJ_y}{l^2} & \frac{4EJ_y}{l} & 0 \\[0.5em]
-\frac{EA}{l} & 0 & \frac{EA}{l}
\end{bmatrix}
\cdot
\begin{Bmatrix} q_3 \\ q_4 \\ q_6 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix}
$$

Eq. III) $$-\frac{EA}{l}\cdot q_3 + 0\cdot q_4 + \frac{EA}{l} q_6 = -P$$

$$q_6 = q_3 - \frac{PL}{EA}$$

Eq II) $$-\frac{6EJ_y}{l^2} q_3 + \frac{4EJ_y}{l}\cdot q_4 + 0\cdot q_6 = 0$$

$$q_4 = \frac{3}{2l}\cdot q_3$$

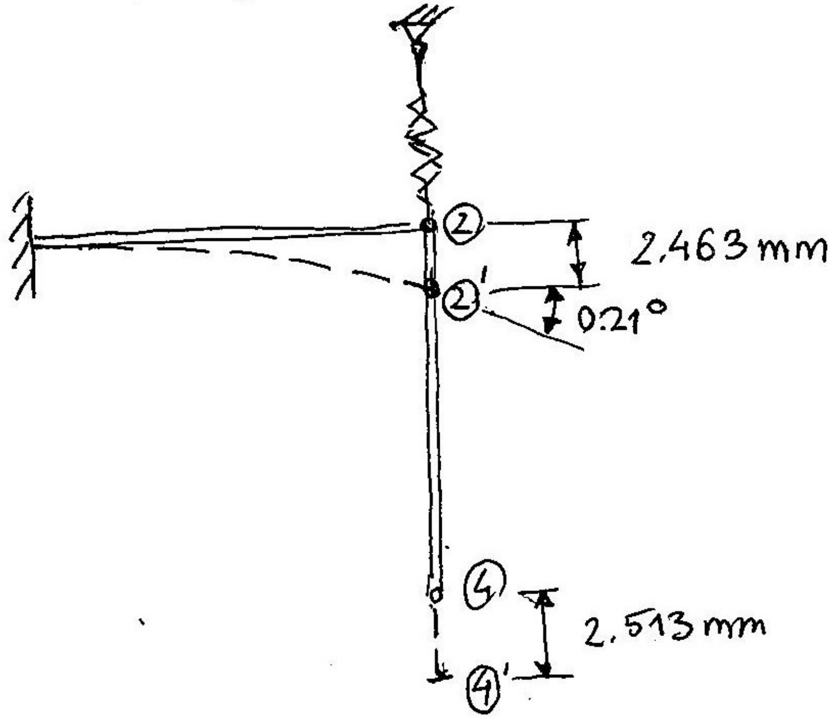
Eq I) $$\left(\frac{12EJ_y}{l^3} + k + \frac{EA}{l}\right)\cdot q_3 - \frac{6EJ_y}{l^2}\cdot q_4 - \frac{EA}{l}\cdot q_6 = 0$$

$$\left(\frac{12EJ_y}{l^3} + k + \frac{EA}{l} - 9\,\frac{EJ_y}{l^3} - \frac{EA}{l}\right) q_3 = -\frac{PL}{EA}\cdot\frac{EA}{l} = -P$$

$$q_3 = -\frac{P}{\frac{3EJ_y}{l^3}+k} = -2.463\,\text{mm}$$

$$q_4 = \frac{3}{2l}\cdot q_3 = -0.00365\,\text{rad} = -0.21^\circ$$

$$q_6 = q_3 - \frac{PL}{EA} = -2.463\,\text{mm} - 0.05\,\text{mm} = -2.513\,\text{mm}$$



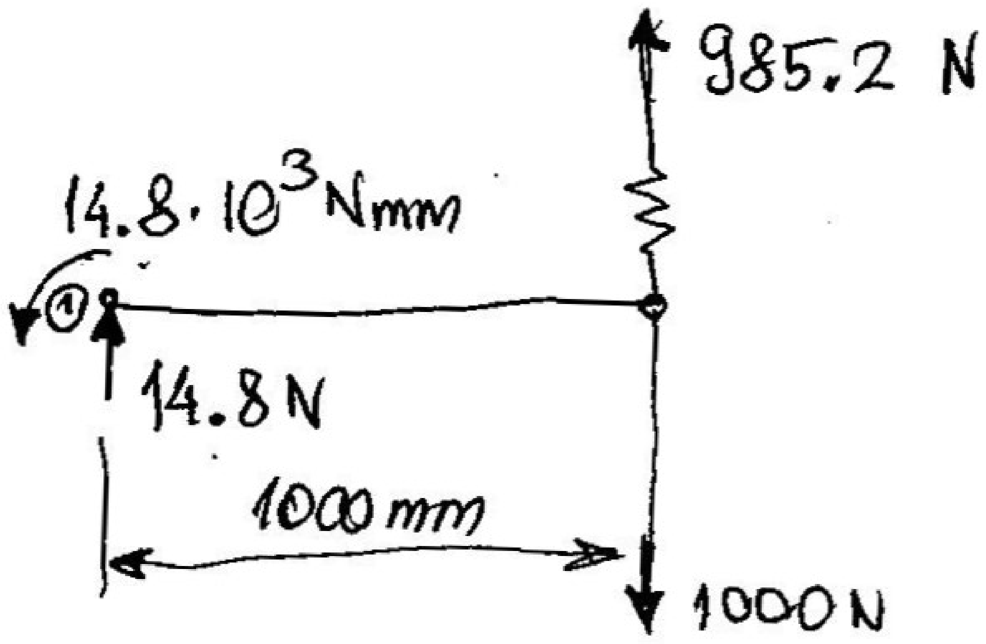
reactions :

$$\underset{6\times 6}{[K]} \cdot \underset{6\times 1}{\{q\}} = \underset{6\times 1}{\{F\}}$$

$$\begin{cases} -\dfrac{12EJ_y}{l^3}\cdot q_3 + \dfrac{6EJ_y}{l^3} q_4 + 0 \cdot q_6 = R_1 \\ -6\dfrac{EJ_y}{l^2} q_3 + \dfrac{2EJ_y}{l} q_4 + 0 \cdot q_6 = M_{R2} \\ -k \cdot q_3 + 0 \cdot q_4 + 0 \cdot q_6 = R_5 \end{cases}$$

$R_1 = 14.8\,N$, $M_{R2} = 14.8 \cdot 10^3\,Nmm$, $R_5 = 985.2\,N$

equilibrium :



$\sum F_z = 0$;

$14.8\,N + 985.2\,N - 1000\,N = 0\,N$

$\sum M_y^{①} = 0$ +↺

$14.8 \cdot 10^3\,Nmm - 1000\,N \cdot 1000\,mm + 985.2\,N \cdot 1000\,mm = 0\,Nmm$

(satisfied)